MATH2048 Honours Linear Algebra II

Midterm Examination 2

Please show all your steps, unless otherwise stated. Answer all **five** questions.

1. Let $T = L_A : \mathbb{R}^4 \to \mathbb{R}^4$ where $A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 1 \end{pmatrix}$. Find all eigenvalues of T and their algebraic multiplicity $\mu_T(\lambda)$ as well as geometric multiplicity $\gamma_T(\lambda)$. Determine whether T is diagonalizable.

   Solution. $f_T(t) = t(t-1)^3$. The distinct eigen values are 0 and 1.

   - For $\lambda_1 = 0$, $A - \lambda_1 I_4 = A$, $\gamma_T(0) = \dim(N(A - \lambda_1 I_4)) = 4 - \text{rank}(A) = 1$. $\mu_T(0) = 1$.
   - For $\lambda_2 = 1$, $A - \lambda_1 I_4 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}$, $\gamma_T(1) = \dim(N(A - \lambda_2 I_4)) = 4 - \text{rank}(A - \lambda_2 I_4) = 2$. But $\mu_T(1) = 3$.

   Hence, T is not diagonalizable.

2. Let $V = M_{2\times 2}(\mathbb{R})$ and $T : V \to V$ be the linear transformation defined by $T(M) = AMB$, where $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$.

   (a) Find a polynomial $g \in P_3(\mathbb{R})$ such that $T^4 = g(T)$.

   (b) Let $M_0 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ and W be the T-cyclic subspace of V generated by M_0. Find $\dim(W)$ and the characteristic polynomial of $T|_W$.

Solution.

(a) Let β be the standard ordred basis for $M_{2\times 2}(\mathbb{R})$. Then $[T]_\beta = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & -1 \end{pmatrix}$ and $f_T(t) = t^2(t+1)^2 = t^4 + 2t^3 + t^2$. By Cayley thm, $f_T(T) = T_0$, i.e. $T^4 + 2T^3 + T^2 = T_0$. Let $g(t) = -2t^3 - t^2$, then $g \in P_3(\mathbb{R})$ and $T^4 = g(T)$.

(b) By computation $M_0 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$, $T(M_0) = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$, $T^2(M_0) = \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}$. Then $T^2(M_0) = -2T(M_0) - M_0$. Therefore, $\{M_0, T(M_0)\}$ forms a basis for W and $\dim(W) = 2$. Since $M_0 + 2T(M_0) + T^2(M_0) = O$, one has $f_{T|_W}(t) = (-1)^2(1 + 2t + t^2)$.

3. Let $T : \mathbb{C}^3 \to \mathbb{C}^3$ be a linear transformation defined by the matrix $A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$, where $\lambda \in \mathbb{C} \setminus \{0\}$ is a nonzero complex number. Find all 1 dimensional and 2 dimensional T-invariant subspaces of $\mathbb{C}^3$.

Solution.

Let W be a T-invariant subspace of V. Then W is $(T - \lambda I)$-invariant too.

For any nonzero $v = (x, y, z)^t \in W$, then $(T - \lambda I)(v) = (y, z, 0)^t \in W$ and $(T - \lambda I)^2(v) = (z, 0, 0)^t \in W$.

- If $\dim(W) = 1$, then $W = \text{span}(\{v\})$. Thus $(y, z, 0)^t = c(x, y, z)^t$ for some $c \in \mathbb{C}$.
  Note that $c \neq 0 \implies z = 0 \implies y = 0 \implies x = 0$ contradicting $v \neq 0$.
  Therefore $c = 0 \implies y = z = 0$. One has $W = \text{span}(\{(x, 0, 0)^t\})$ for $x \in \mathbb{C} \setminus \{0\}$.
  Conversely, if $W = \text{span}(\{(x, 0, 0)^t\})$ for $x \in \mathbb{C} \setminus \{0\}$, then W is 1d T-invariant space.
- If $\dim(W) = 2$. We claim that $z = 0$, otherwise $\{(x, y, z)^t, (y, z, 0)^t, (z, 0, 0)^t\} \subset W$ is linearly independent subset, which implies $\dim(W) \geq 3$. Therefore, $z = 0$ and $W \subset \{(a, b, 0) | a, b \in \mathbb{R}\}$.
  Since $\dim(W) = 2 = \dim(\{(a, b, 0)^t | a, b \in \mathbb{R}\})$, one has $W = \{(a, b, 0)^t | a, b \in \mathbb{R}\} = \text{span}(\{(x, y, 0)^t, (y, 0, 0)^t\})$, where $y \in \mathbb{C} \setminus \{0\}$.
  Conversely, if $W = \text{span}(\{(x, y, 0)^t, (y, 0, 0)^t\})$ for $x \in \mathbb{C}$ and $y \in \mathbb{C} \setminus \{0\}$. Then W is a 2d T-invariant subspace.

4. (a) Let $A = (a_{ij})_{1\leq i,j\leq n} \in M_{n\times n}(\mathbb{C})$, where a_{ij} is the i-th row, j-th column entry of A. Let λ be an eigenvalue of A. Show that:

$$\lambda \in \bigcup_{1\leq i\leq n} \Big\{ z \in \mathbb{C} : |z - a_{ii}| \leq \sum_{1\leq j\leq n, j\neq i} |a_{ij}| \Big\}.$$

(b) Let V be a n-dimensional vector space over $\mathbb{C}$, with an ordered basis $\beta = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$. Given that $n \geq 100$. Consider a linear operator $T : V \to V$ defined by:

$$T(\mathbf{v_1}) = a_1\mathbf{v_1} + b_{11}\mathbf{v_2} + b_{12}\mathbf{v_n}$$
$$T(\mathbf{v_n}) = a_n\mathbf{v_n} + b_{n1}\mathbf{v_1} + b_{n2}\mathbf{v_{n-1}}$$
$$T(\mathbf{v_k}) = a_k\mathbf{v_k} + b_{k1}\mathbf{v_{k+1}} + b_{k2}\mathbf{v_{k-1}} \text{ for } k = 2, 3, ..., n-1.$$

Given that $|a_k| > |b_{k1}| + |b_{k2}|$ for all k. Using (a), show that all eigenvalues of T are non-zero.

Solution.

(a) λ is an eigenvalue of A, so $Ax = \lambda x$ for some nonzero $x \in \mathbb{C}^n$. Let $x = (x_1, ..., x_n)^t$. Find i such that the element of x with the largest absolute value is x_i. Then $x_i \neq 0$ since $x \neq 0$.

Taking the i-th component of the equation $Ax = \lambda x$, one has $\sum_{j=0}^{n} a_{ij}x_j = \lambda x_i$. So $\sum_{j\neq i} a_{ij}x_j = (\lambda - a_{ii})x_i$.

By triangle inequality, $|\lambda - a_{ii}| = |\sum_{j\neq i} a_{ij}\frac{x_j}{x_i}| \leq \sum_{j\neq i} |a_{ij}|$ since $|\dfrac{x_j}{x_i}| \leq 1$.

(b) If λ is an eigenvalue of T, then λ is an eigenvalue of $[T]_\beta$ and thus is an eigenvalue of $[T]_\beta^t$, the transpose of $[T]_\beta$.

By (a), $|a_k| - |\lambda| \leq |\lambda - a_k| \leq |b_{k1}| + |b_{k2}| < |a_k|$, so $|\lambda| > 0$, which implies all eigenvalues of T are non-zero.

5. Let $T : V \to W$ be a linear transformation between the vector spaces V and W over $\mathbb{C}$. Let T^* be the transpose of T. Prove or disprove that $(W/R(T))^*$ is isomorphic to $N(T^*)$. If it is, please construct an isomorphism between $(W/R(T))^*$ and $N(T^*)$. If it is not, please give a rigorous proof. Please explain your answer with details. (Here, $(W/R(T))^*$ is the dual space of $(W/R(T))$.)

Solution. Consider

$$\begin{aligned} \Phi : (W/R(T))^* &\to N(T^*) \\ h &\mapsto \Phi(h) \end{aligned}$$

defined by $\Phi(h)(w) = h(w + R(T))$ for all $h \in (W/R(T))^*$ and $w \in W$. We show that Φ is an isomorphism.

- Well-defined.

  For any $h \in (W/R(T))^*$, $T^*(\Phi(h))(v) = (\Phi(h) \circ T)(v) = \Phi(h)(T(v)) = \Phi(T(v) + R(T)) = h(R(T)) = 0$ for any $v \in V$. Therefore $\Phi(h) \in N(T^*)$ for any $h \in (W/R(T))^*$.

  Besides, if $\Phi(h_1) \neq \Phi(h_2)$, there exists $w \in W$ such that $\Phi(h_1)(w) \neq \Phi(h_2)(w)$, i.e. $h_1(w + R(T)) \neq h_2(w + R(T))$ So $h_1 \neq h_2$.
- Linear. It's trivial.
- Injective.

  For any $h \in N(\Phi)$, $\Phi(h) = 0$. That is $h(w + R(T)) = \Phi(h)(w) = 0$ for all $w \in W$. So $h = 0$.
- Surjective.

  For any $g \in N(T^*)$, define h by $h(w + R(T)) = g(w)$.

  For any $w_1 + R(T) = w_2 + R(T)$, one has $w_1 - w_2 \in R(T)$. There exists $v_0 \in V$ such that $w_1 - w_2 = T(v_0)$ and $g(w_1 - w_2) = g(T(v_0)) = T^*(g)(v_0) = 0$ for $g \in N(T^*)$.

  Therefore, $h(w_1 + R(T)) = g(w_1) = g(w_2) = h(w_2 + R(T))$, then h is well-defined. Since $h \in (W/R(T))^*$ and $\Phi(h) = g$, one has Φ is surjective.